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On the Numerical Solution of Two-Dimensional Elasticity Problems

FRANCES BAUER AND EDWARD L. REISS

Courant Institute of Mathematical Sciences, New York University, New York, New York 10003

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An iterative method is presented for solving plane strain and plane stress problems for homogeneous and isotropic elastic materials. Displacements or some combination of displacements and stresses are prescribed on the boundary of the elastic solid. The iterates are evaluated numerically by difference methods. The direct block factoring method is used to solve the resulting system of algebraic equations. Applications to specific problems are given. A proof of the convergence of the analytic iterations is given for problems where the displacements are specified on the entire boundary.

1. INTRODUCTION AND FORMULATION

We present an iterative method for solving plane stress and plane strain problems for homogeneous isotropic elastic materials. We consider a bounded region D in the x, y plane, which may be multiply connected. The boundary of D is denoted by B. The arc length along B is s and the outward unit normal and tangent vectors to B are $\mathbf{n}(s) = (n_1, n_2)$ and $\mathbf{t}(s) = (t_1, t_2)$, respectively. Stresses and/or displacements are prescribed on the boundary B of the elastic body. We wish to determine the resulting displacements and stresses in the interior.

The biharmonic boundary value problem,

$$\Delta^2\phi = p(x, y), \qquad \text{for } x, y \text{ in } D, \tag{1.1a}$$

$$\phi = f(s), \quad \mathbf{n}\cdot\nabla\phi = g(s), \qquad \text{for } x, y \text{ on } B, \tag{1.1b}$$

is the conventional mathematical formulation of plane strain and plane stress problems if only stresses are prescribed on B. In (1.1) Δ is the two dimensional Laplacian, ϕ is the Airy stress function and p, f, and g are prescribed functions that are determined by the applied forces.

In this paper we shall consider plane stress and plane strain problems where the displacements or some mixture of displacements and stresses are specified on B. For these problems, the formulation (1.1) may be awkward. Then it is convenient to employ the displacement formulation of plane stress and plane strain. Thus we consider the displacement vector $\mathbf{u}(x, y) = [u(x, y), v(x, y)]$. The displacement equations of equilibrium are

$$\Delta u = k(u_y - v_x)_y, \qquad \Delta v = -k(u_y - v_x)_x \tag{1.2a}$$

where k is defined by

$$2k \equiv \begin{cases} 1 + \nu, & \text{for plane stress,} \\ (1 - \nu)^{-1}, & \text{for plane strain,} \end{cases} \tag{1.2b}$$

and ν is Poisson's ratio. The boundary conditions are that at each point of B either the normal displacement or the normal stress and either the tangential displacement or the tangential stress are prescribed. That is,

$$\text{either} \quad \mathbf{n} \cdot \mathbf{u} = D_n(s) \qquad \text{or} \qquad \tau_{ij} n_i n_j = N(s), \tag{1.2c}$$

and

$$\text{either} \quad \mathbf{t} \cdot \mathbf{u} = D_t(s) \qquad \text{or} \qquad \tau_{ij} t_i n_j = T(s), \tag{1.2d}$$

are prescribed at each point of B. The stresses $\tau_{ij} = (\tau_x, \tau_y, \tau_{xy})$ are related to the displacement gradients by Hooke's law,

$$\tau_x = \alpha(u_x + \beta v_y), \qquad \tau_y = \alpha(v_y + \beta u_x), \qquad \tau_{xy} = \frac{E}{2(1 + \nu)}(u_y + v_x), \tag{1.3a}$$

where E is Young's modulus and α and β are defined by

$$[\alpha, \beta] = \begin{cases} \left[\dfrac{E}{1 - \nu^2}, \nu\right], & \text{for plane stress,} \\ \left[\dfrac{(1 - \nu) E}{(1 + \nu)(1 - 2\nu)}, \dfrac{\nu}{1 - \nu}\right], & \text{for plane strain.} \end{cases} \tag{1.3b}$$

We shall assume, in all the problems that we consider, that there is an arc of B on which either D_n and/or D_t are prescribed. If N and T were prescribed on all of B, then we would employ the formulation (1.1). Prescribing D_n and D_t on an arc B_u of B is equivalent to prescribing $\mathbf{u}$ on B_u. If $B_u \equiv B$, then we call (1.2) the elasticity Dirichlet problem. If $B_u \not\equiv B$, then we call (1.2) a mixed problem.

Problem (1.1) is also the conventional mathematical formulation of the classical Lagrange–Kirchhoff small deflection theory of plates where ϕ is the displacement of the plate perpendicular to the midplane. The boundary conditions (1.1b) imply that the displacement and slope are specified on the edge of the plate. The formulation (1.1) may be inconvenient for numerical computations for other boundary conditions, such as specifying the moment and shear force on the edge. Southwell

[1] has introduced an alternative formulation of the Lagrange–Kirchhoff theory in which u and v are "moment potentials" that satisfy (1.2a) with $k = (1 - \nu)/(1 + \nu)$. The moment and shear boundary conditions are then equivalent to specifying u and v on B. Thus the method presented in this paper is applicable to a class of plate bending problems (see e.g. [2]).

We obtain approximate solutions of (1.2) by an accelerated iteration method. Each iterate is the solution of a boundary value problem for Poisson's equation. The iterates are then approximated by solving the Poisson boundary value problem numerically. We establish the convergence of the iterates to the solution of (1.2) for the elasticity Dirichlet problem. Applications of the method to Dirichlet and mixed problems are described in Section 4. The numerical results suggest that the iterations converge for mixed problems.

Special applications of the method are given in [2–4]. The method is related to the iterative procedures previously used for the numerical solution of nonlinear plate and shell problems (see e.g. [5] and references given therein).

2. The Iterative Method

We shall describe the iterative method for the elasticity Dirichlet problem. Typical modifications that are necessary to treat mixed problems are discussed in Section 4. Thus we wish to solve (1.2a) subject to the boundary conditions,

$$\mathbf{u}(x, y) = \mathbf{f}(s), \qquad \text{for} \quad x, y \text{ on } B, \tag{2.1}$$

where $\mathbf{f}$ is a prescribed vector function.

Starting from an initial estimate $\mathbf{u}^0(x, y)$ of the solution, we define a sequence of iterates $\mathbf{u}^n(x, y)$ by the recursions,

$$\begin{aligned} \Delta \bar{u}^n &= k(u_y^{n-1} - v_x^{n-1})_y, \\ \Delta \bar{v}^n &= -k(u_y^{n-1} - v_x^{n-1})_x, \qquad \text{for } x, y \text{ in } D, \end{aligned} \tag{2.2a}$$

$$\bar{\mathbf{u}}^n(x, y) = \mathbf{f}(s), \qquad \text{for } x, y \text{ on } B, \tag{2.2b}$$

$$\mathbf{u}^n = \theta \bar{\mathbf{u}}^n + (1 - \theta)\, \mathbf{u}^{n-1}, \qquad \text{for } x, y \text{ in } D. \tag{2.2c}$$

In (2.2), $\bar{\mathbf{u}}^n$ is a provisional iterate and the number θ is the acceleration parameter. If $\theta = 1$, then (2.2) are simple iterations.

Each iterate in (2.2) is the solution of the Poisson boundary value problem,

$$\begin{aligned} \Delta w &= H(x, y), \qquad \text{for} \quad x, y \text{ in } D, \\ w &= F(s), \qquad \text{for} \quad x, y \text{ on } B. \end{aligned} \tag{2.3}$$

At each step of the iterations H and F are determined from the previous iterates and the data.

We now show that the simple iterations converge for sufficiently smooth data. Applications of the method, some of which are described in Section 4, show that the rate of convergence can be improved considerably by choosing other values of θ. We use the conventional notation $C_{m+\alpha}(D)$ for the space of functions defined on D whose mth derivatives are Hölder continuous with positive exponent $\alpha < 1$ in $D + B$. We require that the boundary B and the boundary data $\mathbf{f}$ are sufficiently smooth so that, the elasticity Dirichlet problem and (2.3) have unique solutions, the iterations (2.2) are defined and the divergence theorem is applicable to D. Thus we assume that: *B is of class $C_{2+\alpha}$; the initial iterate belongs to $C_{2+\alpha}(D)$; $\mathbf{f}$ is continuous and $d\mathbf{f}/ds$ is piecewise continuous.* It follows from the existence and regularity theory for second order strongly elliptic equations, that the iterations (2.2) are defined under the above conditions [6]. The results of this section can also be established with weaker restrictions.

If $\mathbf{u}$ is the solution of the elasticity Dirichlet problem, we define $\mathbf{U}^n(x, y) = [U^n(x, y), V^n(x, y)]$ by

$$\mathbf{U}^n \equiv \mathbf{u} - \mathbf{u}^n. \tag{2.4}$$

Then we conclude from (1.2a), (2.1), and (2.2) that $\mathbf{U}^n$ is a solution of

$$\Delta U^n = k W_y^{n-1}, \tag{2.5a}$$

$$\Delta V^n = -k W_x^{n-1}, \tag{2.5b}$$

$$\mathbf{U}^n = 0, \qquad \text{for } x, y \text{ on } B, \tag{2.5c}$$

where $W^n(x, y)$ is defined by

$$W^n \equiv U_y{}^n - V_x{}^n. \tag{2.5d}$$

We shall denote the L_2 norm by $\| \ \|$, i.e.,

$$\| z(x, y)\|^2 \equiv \iint z^2 \, dx \, dy. \tag{2.6}$$

In (2.6), and throughout this section, the integration is over the region D. Our convergence result is stated in the following theorem.

THEOREM. *$\mathbf{U}^n$ and its first partial derivatives converge to zero in the L_2 norm as $n \to \infty$.*

Proof. We multiply (2.5a) by U^n and (2.5b) by V^n and add the resulting equations. Then we use Green's theorem and (2.5c). This yields,

$$Z_n^2 \equiv \iint [(\nabla U^n)^2 + (\nabla V^n)^2]\, dx\, dy = k \iint (-U^n W_y^{n-1} + V^n W_x^{n-1})\, dx\, dy. \tag{2.7}$$

We insert the identity,

$$-U^n W_y^{n-1} + V^n W_x^{n-1} = (V^n W^{n-1})_x - (U^n W^{n-1})_y + W^n W^{n-1}$$

into the right side of (2.7). Then by applying the divergence theorem to the result and using (2.5c), we obtain

$$Z_n^2 = k \iint W^n W^{n-1}\, dx\, dy. \tag{2.8}$$

We estimate the right side of (2.8) by observing that

$$\begin{aligned}\iint (W^n)^2\, dx\, dy &\leqslant \iint [(W^n)^2 + (U_x^n + V_y^n)^2]\, dx\, dy \\ &= \iint (\nabla U^n)^2 + (\nabla V^n)^2\, dx\, dy + 2 \iint (U_x^n V_y^n - U_y^n V_x^n)\, dx\, dy.\end{aligned} \tag{2.9}$$

The last integral on the right side vanishes because of (2.5c). Thus (2.9) is reduced to

$$\iint (W^n)^2\, dx\, dy \leqslant Z_n^2. \tag{2.10}$$

We now apply Schwarz's inequality to the right side of (2.8) and then use (2.10) in the result. This gives,

$$Z_n^2 \leqslant k Z_n Z_{n-1}\,. \tag{2.11}$$

Since ν must be in the interval $0 \leqslant \nu < \frac{1}{2}$, (1.2b) implies that $k < 1$ for both plane strain and plane stress problems. Thus we conclude from (2.11) that

$$\lim_{n \to \infty} Z_n = 0. \tag{2.12}$$

It follows from (2.12) and (2.7) that the L_2 norms of the first partial derivatives of U^n and V^n converge to zero as $n \to \infty$.

To complete the proof, we use Poincaré's inequality. Thus there is a constant $C > 0$ such that for any $g(x, y) \in C_2(D)$,

$$\| g \| \leqslant C \| \nabla g \|. \tag{2.13}$$

Since the norms of the first partial derivatives of U^n and V^n converge to zero, we conclude by inserting $g = U^n$ and $g = V^n$ in (2.13) that

$$\lim_{n\to\infty} \| \mathbf{U}^n \| = 0,$$

which establishes the theorem.

Thus the iterates that are defined by (2.2) and their first derivatives converge in the L_2 norm to the solution of the elasticity Dirichlet problem. The numerical results presented in Section 4 suggest that the theorem is valid for mixed problems. However, we have not yet been able to prove this result. By suitable modifications in the proof of the theorem, we can establish a constructive existence theorem for elasticity Dirichlet problems in two and three dimensions.

3. Numerical Methods

We evaluate the iterates in (2.2) by obtaining numerical solutions of the boundary value problem (2.3). Conventional finite difference approximations of (2.3) or the finite element method yield a system of algebraic equations

$$M\mathbf{w} = \mathbf{r}. \tag{3.1}$$

The vector $\mathbf{w}$ is obtained by appropriately ordering the elements of the mesh function w_{ij} (or the nodal displacements) that approximate the solution $w(x_i, y_j)$ of (2.3) on the mesh (or at the nodes). For each step of the iteration procedure $\mathbf{r}$ is a known vector that is determined by the data and the previous iterates.

If the standard five point or nine point difference approximations of the Laplacian are used, then the matrix M is of the block tridiagonal form,

$$\begin{aligned} M &= [A_i, B_i, C_i] \\ &= \begin{pmatrix} B_1 & C_1 & 0 & \cdot & \cdot & \cdot & 0 \\ A_2 & B_2 & C_2 & 0 & \cdot & \cdot & 0 \\ 0 & A_3 & B_3 & C_2 & 0 & \cdot & 0 \\ \vdots & 0 & \cdot & \cdot & \cdot & & \\ 0 & \cdots & & 0 & A_{q-1} & B_{q-1} & C_{q-1} \\ 0 & \cdots & & & 0 & A_q & B_q \end{pmatrix}, \end{aligned} \tag{3.2}$$

where A_i, B_i, C_i are matrices. Each of the diagonal matrices B_i are square. The dimensions of the other matrices in (3.2) are consistent with the indicated partitioning. The algebraic equations that are obtained from other difference approximations or the finite element method may also be of block diagonal form with possibly a larger number of bands.

A fast method must be used to solve (3.1) because we wish to solve large systems and many iterations may be required for convergence. We employed either the five or the nine point difference approximations to the Laplacian in all the problems studied. Then the matrix M is of block tridiagonal form. We used the direct block factoring method [7] to solve (3.1).

In this method the matrix M is factored into the product of an upper block triangular matrix U and a lower block triangular matrix L. That is,

$$M = LU, \tag{3.3}$$

where, using the notation given in (3.2), L and U are defined by

$$\begin{aligned} L &= [A_i, \beta_i, O], \\ U &= [O, I_i, \gamma_i]. \end{aligned} \tag{3.4}$$

In (3.4), I_i are unit matrices of the same dimension as B_i, and β_i and γ_i are matrices that are defined for $i = 1, 2,..., q$, by

$$\begin{aligned} \beta_i &= B_i - A_i\gamma_{i-1}, \qquad \gamma_0 \equiv 0 \\ \gamma_i &= \beta_i^{-1}C_i, \end{aligned} \tag{3.5}$$

where we define $\gamma_0 \equiv 0$. Then by using (3.3), the system (3.1) is equivalent to the two systems,

$$L\mathbf{v} = \mathbf{r}, \tag{3.6}$$

$$U\mathbf{w} = \mathbf{v}. \tag{3.7}$$

The systems (3.6) and (3.7) are of block triangular form. Therefore they can be solved directly. First we partition $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{w}$ into subvectors to conform with the partitioning of L and U. Then the solutions of (3.6) and (3.7) are recursively given by

$$\begin{aligned} \mathbf{v}_i &= \beta_i^{-1}(\mathbf{r}_i - A_i\mathbf{v}_{i-1}), \qquad i = 1, 2,..., q, \\ \mathbf{w}_i &= \mathbf{v}_i - \gamma_i\mathbf{w}_{i+1}, \qquad i = q, q-1,..., 1. \end{aligned} \tag{3.8}$$

Thus the matrices $A_i, \gamma_i, \beta_i^{-1}$ are needed to evaluate the solution of (3.1) by the formulas (3.8). The submatrices A_i, B_i and C_i are usually sparse. However for $i > 1$, β_i and γ_i are not sparse. The inverses β_i^{-1} are evaluated by Gauss elimination with pivotal condensation. Since M does not change from step to step in the iteration procedure, the β_i^{-1} and γ_i matrices are computed only once. They are stored in the fast access memory of the computer to achieve greater speed of computation. However, this storage limits the size of the algebraic systems that we can consider. The maximum size that we used in our computations was approximately 1200 equations, although slightly larger systems can be accommodated. If auxiliary memory devices, such as tapes and discs are employed, then significantly larger systems can be treated but the speed of computation is then seriously reduced. The advantages of this method are its speed, once the factoring is completed, and its applicability to a variety of domains.[1]

For rectangular or simple regions composed of parallel rectangles, block reduction and fast Fourier transform methods, e.g. [8, 9], may be applicable. Then the size of the fast access storage is significantly reduced and more refined meshes can be used. For the mesh sizes that we used (approximately 600–1200 points), the factoring and fast Fourier methods are comparable in speed.

If M is of block five diagonal form, then the factoring and block reduction method presented in [10] can be employed to solve (3.1).

The following numerical convergence criterion was employed,

$$\max_{(x_i,y_j)\subset\text{mesh}} | \mathbf{u}_{ij}^n - \mathbf{u}_{ij}^{n-1} | < 10^{-a}. \tag{3.9}$$

In most of our calculations we used $a = 8$. Only small changes in the answers occurred when larger values of a were employed. The number of iterations that are required to satisfy (3.9) depends on the domain, the boundary conditions, the mesh width and the value of θ.

When the numerical iterations satisfied the convergence criterion (3.9), numerical approximations to the stresses at each point of the mesh were computed from difference approximations to (1.3a).

4. Applications of the Method

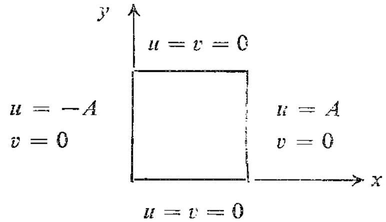
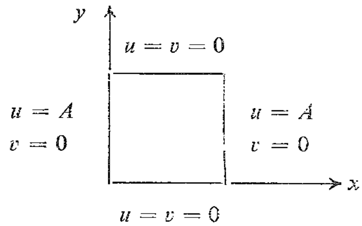
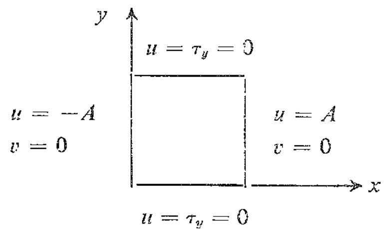
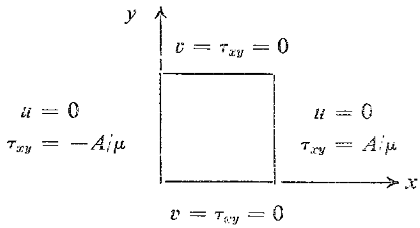
The method was applied to a variety of problems. Dirichlet and mixed problems were considered. We shall briefly describe some of the numerical results for four of the problems that we studied. In each problem D is the unit square. The boundary conditions are summarized in Column 2 of Table I. In Table I, A and μ are defined by

$$A(y) \equiv 4y(1-y), \qquad \mu \equiv \frac{E}{2(1+\nu)}. \tag{4.1}$$

More general polygonal regions and regions with curved boundaries were also considered (see e.g. [2,4]).

[1] If the domain or the coefficients in (1.2a) vary, then the size and elements of the blocks in the corresponding matrix M will vary.

TABLE I

Problem no.	Boundary conditions	$\theta_c \approx$	N
I		1.42	22
II		1.42	22
III		1.8	133
IV		1.9	159

Problems I and II are Dirichlet problems. Since Problems III and IV are mixed problems, it is necessary to modify the iteration procedure (2.2) for them. We observe that τ_y is given by (1.3). Therefore, we replace the boundary condition (2.2b) on $y = 0, 1$ for Problem III by

$$u^n = 0, \qquad v_y{}^n = -\beta u_x^{n-1} = 0. \tag{4.2}$$

Similarly, for Problem IV, the boundary conditions for the iteration method are

$$\begin{aligned} u^n &= 0, \qquad v_x{}^n = -u_y^{n-1} + A(y) = A, \qquad && x = 0, 1, \\ v^n &= 0, \qquad u_y{}^n = 0, && y = 0, 1. \end{aligned} \tag{4.3}$$

For more complicated mixed problems, the modified iteration procedures require iteration in the boundary conditions.

In all the problems we used $\nu = 0.32$ and the plane stress values for α, β, and k, see (1.2b) and (1.3b). Furthermore we used

$$\mathbf{u}^0 \equiv 0 \qquad \text{and} \qquad \delta = 1/26 \tag{4.4}$$

in all the computations where δ is the mesh width in the difference approximation.

Finer meshes can be accommodated with our method. For a mesh with $\delta = 1/26$ we determined the unit time T for a single iteration on the CDC 6600 computer as $T = 0.206$ sec.

We wish to select the acceleration parameter θ in (2.2) so as to minimize the number of iterations necessary to satisfy (3.9). We denote these optimum values as θ_c. Estimates of θ_c were obtained from numerical experiments. The results are given in the third column of Table I. The values of θ_c depend upon the domain D and the type of boundary conditions. In Column 4 of Table I we list the number N of iterations that are required to satisfy (3.9) with the values of θ given in Column 3. The total iteration time required to solve each problem is equal to $TN \approx .2N$. We observe that N increases significantly as the arc of the boundary on which $\mathbf{u}$ is prescribed decreases. The iterations converge in each case with $\theta = 1$. For example, for Problem I, 31 iterations were required for convergence with $\theta = 1$ and for Problem III, approximately 250 iterations were required with $\theta = 1$. The number of iterations necessary for convergence can be reduced by taking more accurate initial iterates $\mathbf{u}^0$ and by decreasing the value of a in (3.9).

In Fig. 1, we present computer drawn sketches of the numerically determined displacements and stresses for each of the problems.

For $\nu = 0.32$, the plane stress value of k as given by (1.2b) is $k = 0.66$. Since 31 iterations were required for Problem I with $\theta = 1$, the convergence rate in the L_2 norm is, see (2.11), $k^{31} \approx (0.66)^{31} \approx 10^{-5.6}$. This is reasonably close to the convergence criterion of 10^{-8}, see (3.9), which is essentially in the maximum norm.

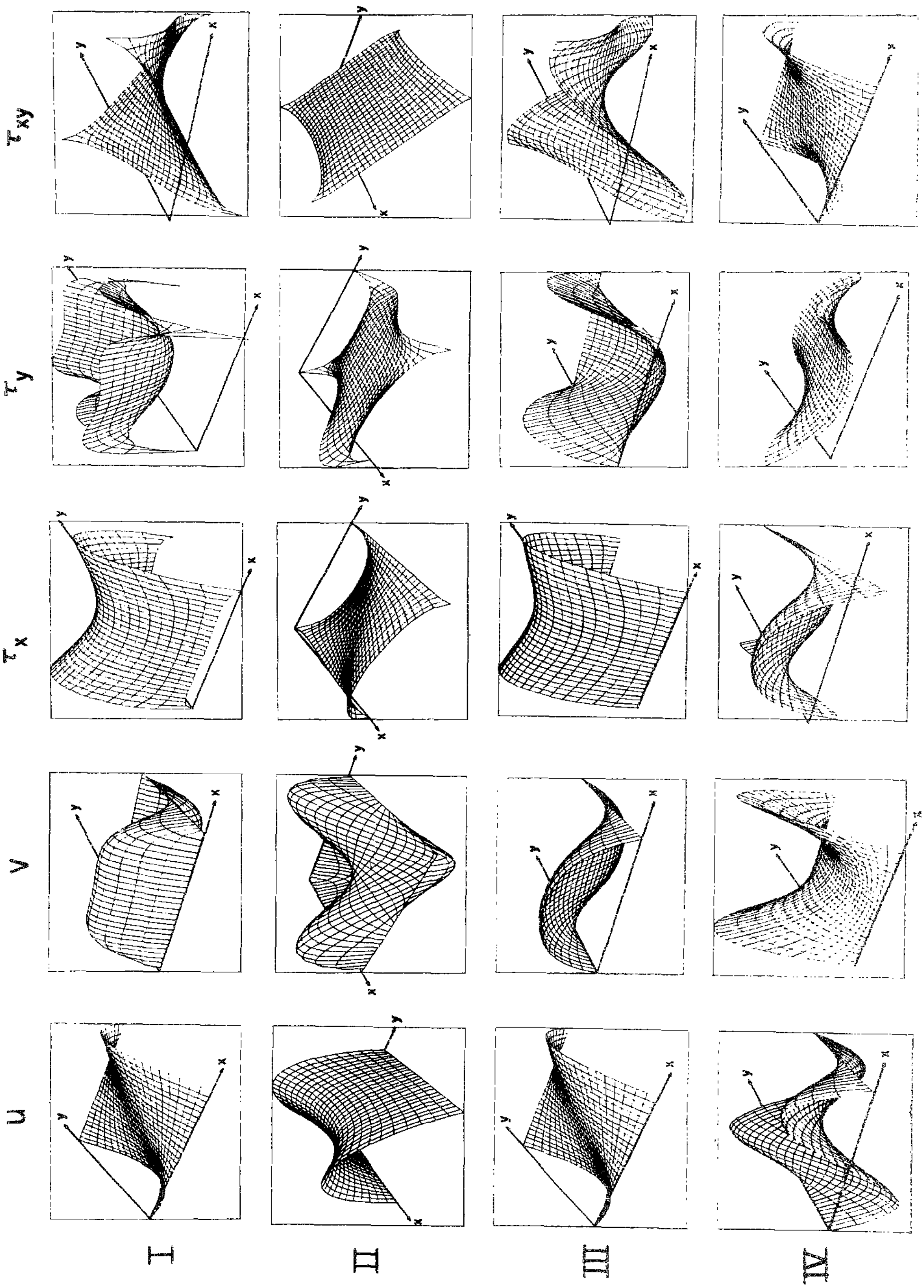


FIG. 1. Displacements and stresses for Problems I–IV.

Poisson's ratio is in the range, $0 \leqslant \nu < \frac{1}{2}$. For $\nu = \frac{1}{2}$, the elastic material is incompressible and the formulation must be modified. For $\nu = \frac{1}{2}$ we have from (1.2b)

$$k = \begin{cases} 3/4, & \text{for plane stress,} \\ 1, & \text{for plane strain.} \end{cases}$$

The analysis in Section 2 shows that the iterations converge with $\theta = 1$ for all ν in $0 \leqslant \nu < \frac{1}{2}$. Since $k(\nu)$ is a monotone increasing function, the analysis suggests that the number of iterations required for convergence increases as $\nu \to \frac{1}{2}$. We solved Problem I with $\nu = 0.499$. Thirty one iterations were required for convergence for the plane stress problem. For the plane strain problem, $k(0.499) = 0.998$ and it was possible to satisfy (3.9) with $a = 6$ after 625 iterations. Presumably many more iterations are necessary to satisfy (3.9) with $a = 8$.

5. Concluding Remarks

A variety of other iteration procedures can be defined for (1.2). For example, we can rewrite (1.2a) and hence (2.2a) as

$$\Delta\bar{u}^n = -k_0(u_x^{n-1} + v_y^{n-1})_x\,, \qquad \Delta\bar{v}^n = -k_0(u_x^{n-1} + v_y^{n-1})_y\,, \tag{5.1}$$

where k_0 is defined by

$$k_0 = \frac{k}{1-k} = \begin{cases} (1+\nu)/(1-\nu), & \text{for plane stress,} \\ 1/(1-2\nu), & \text{for plane strain.} \end{cases}$$

Since $k_0(\nu)$ is a monotonically increasing function and $k_0(0) = 1$, the analysis in Section 2 suggests that the simple iterations in (5.1) may diverge. This was confirmed by numerical experiments. However convergence was obtained with appropriate values of θ. The convergence of the iterations (2.2) was always faster than the iterations (5.1).

A finite element formulation proposed by Rashid [11] is equivalent to the iterations

$$\Delta\bar{u}^n - k\bar{u}_{yy}^n = -kv_{xy}^{n-1}, \qquad \Delta\bar{v}^n - k\bar{v}_{xx}^n = -ku_{xy}^{n-1}. \tag{5.2}$$

In the elasticity Dirichlet problem the submatrices in M corresponding to the first and second equations in (5.2) are different. This increases the storage requirement. Furthermore the matrix M and its factors L and U must be recomputed for each choice of ν because k depends on ν.

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